

On the energy of homogeneous cosmologies

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An energy for the homogeneous cosmological models is presented. More specifically, using an appropriate natural prescription, we find the energy within any region with any gravitational source for a large class of gravity theories—namely those with a tetrad description—for all 9 Bianchi types. Our energy is given by the value of the Hamiltonian with homogeneous boundary conditions; this value vanishes for all regions in all Bianchi class A models, and it does not vanish for any class B model. This is so not only for Einstein's general relativity but, moreover, for the whole 3-parameter class of tetrad-teleparallel theories. For the physically favored one parameter subclass, which includes the teleparallel equivalent of Einstein's theory as an important special case, the energy for all class B models is, contrary to expectation, *negative*.

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I. INTRODUCTION

Gravity is the only universal force; it is long range and dominates the cosmos. Energy has been one of the most useful physical concepts, no less so in gravitating systems—one need only recall its utility in the Newtonian Kepler problem. From the modern relativistic perspective energy-momentum is the source of gravity, yet—somewhat ironically—unlike all matter and other interaction fields in the absence of gravity, identifying a good description of the energy-momentum of gravitating systems has been the oldest and most controversial outstanding puzzle. Physically the fundamental difficulty can be understood as a consequence of Einstein's equivalence principle—from which it follows that gravity cannot be detected at a point. Hence the energy-momentum of gravity cannot have a proper local density; it is fundamentally non-local and thus inherently non-tensorial and therefore *non-covariant*. (For a good discussion of this point see [1].)

After Einstein proposed his gravitational energy-momentum density (see e.g. [2]), various alternate prescriptions were introduced by other researchers (notably Papapetrou [3], Bergman and Thompson [4], Møller [5], Goldberg [6], Landau and Lifshitz [7], and Weinberg [8]). These investigations lead to a variety of expressions with no compelling criteria for favoring any particular one. Moreover these traditional energy-momentum *pseudotensors* are, as noted, necessarily not covariant objects, they inherently depend on the coordinates, so they cannot provide a truly physical local gravitational energy-momentum density. Caught between the equivalence

principle and the covariance principle, the pseudotensor approach has been largely questioned, although never completely abandoned.

On the other hand, because the gravitational interaction is local, some kind of local—or at least nearly local—description of the associated energy-momentum was still sought. The modern concept, introduced by Penrose [9] to resolve this dilemma, is that properly energy-momentum is *quasi-local*, being associated with a closed surface bounding a region (for a nice review of the topic see [10]). However it soon became clear that there is no unique quasi-local energy expression. Although there are some especially famous ones (e.g., Brown-York [11] and Liu-Yau [12]) many definitions of quasi-local mass-energy have been proposed; they generally give distinct results. For example Bergqvist [13] studied several different quasi-local mass definitions for the Kerr and Reissner-Nordström spacetimes and came to the conclusion that not even two of the examined definitions gave the same result.

Our view is that from the Hamiltonian perspective one can make sense of this situation—understanding not only why all these otherwise perplexing choices exist but also what is their real physical significance. Simply put the energy of a gravitating system within a region—regarded as the value of the Hamiltonian for this system—naturally depends not only on the interior of the region but also on the *boundary conditions* imposed at the interface with the exterior (this is reasonable, after all the particular solution to the field equations depends on the boundary conditions). It has been found that the Hamiltonian necessarily includes an integral over the 2-surface bounding the considered region. This Hamiltonian boundary term plays two key roles: (i) it controls the value of the Hamiltonian, and (ii) its specific form (via the requirement that the boundary term in the variation in the Hamiltonian vanish) is directly re-

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lated to the selected type and value of the boundary conditions. Many of the quasi-local proposals (namely all those which admit a Hamiltonian representation) can be understood in these terms: their differences are simply associated with different boundary conditions. Furthermore, using a covariant Hamiltonian formalism, it has been shown [14, 15] that every energy-momentum pseudotensor can be associated with a particular Hamiltonian boundary term—which in turn determines the quasi-local energy-momentum that is respectively linked with the implied boundary conditions. In this sense, it has been said that the Hamiltonian quasi-local energy-momentum approach rehabilitates the pseudotensors and, moreover, dispels any doubts about the physical meaning of these energy-momentum complexes—since with this approach all their inherent ambiguities are given clear physical and geometric meanings.

We want to emphasize that while there are many possible Hamiltonian boundary term energy-momentum quasi-local expressions—simply because there are many conceivable boundary conditions—it nevertheless has been found [16] that in practice there usually is a particular choice *best suited* to the task at hand. It has long been known that this is the case for familiar physical systems where gravity is negligible. For example in thermodynamics we have not a unique energy but rather the internal energy, the enthalpy, and the Helmholtz and Gibbs free energies—each a real physical energy adapted to a specific interface between the physical system of interest and its surroundings; similarly in classical electrostatics the physically appropriate measure of energy obtained from the work-energy relation for a finite system depends on whether one considers fixed surface charge density or fixed potential—boundary condition choices which are respectively associated with the symmetric and the canonical energy-momentum tensor.

Here we apply this Hamiltonian boundary term quasi-local energy-momentum approach to homogeneous cosmologies. Our motivation is twofold. On the one hand the utility of having a good measure of the energy of such gravitating systems—a measure with sensible answers for the ideal exactly homogeneous case but which can be applied to any perturbations thereof and even quite generally—should be obvious. We are likewise motivated by the consideration that homogeneous cosmology affords an excellent set of models where one can test the suitability of our—or indeed any other—proposed (quasi-)local energy-momentum ideas.

Accordingly it is important to here remark on our specific choice of energy expression for these homogeneous cosmologies. We have noted five different approaches which lead us to exactly the same formula for energy-momentum for the homogeneous cosmologies in General Relativity (GR). (i) One may take the tetrad form of GR: among the many traditional energy-momentum approaches Møller’s [17] tetrad-teleparallel formulation has been highly regarded. This form of GR can be viewed from the gauge theory perspective, then a certain *trans-*

lational gauge current (closely related to the energy-momentum expression proposed by Møller) stands out (among other virtues it is the only classical expression that has an associated *positive energy* proof [18]). (ii) Another perspective is via the GR Hamiltonian: from the covariant Hamiltonian approach for GR one particular Hamiltonian boundary term stands out as being the favored expression for general applications [16]. In many situations, including the present application, it reduces to the tetrad-teleparallel gauge current expression. (iii) Alternately one can proceed from the spinor-parameterized Hamiltonian associated with the Witten positive energy proof [19, 20, 21]. For an isolated asymptotically flat gravitating system the spinor field may be taken to satisfy Witten’s equation, then the Hamiltonian has a positive value; however for a *homogeneous* cosmology the Witten equation does not have appropriate solutions, instead it is appropriate to take the spinor field to be *homogeneous*, which reduces the Hamiltonian to the aforementioned expression. (iv) A fourth approach—which we like for its simplicity, generality and straightforwardness—is to consider the general teleparallel theory (which includes Einstein’s GR as a special case). For such theories, quite unlike the GR situation, investigators have advocated only two specific expression for the energy-momentum density (this is one of the virtues of treating this whole class of theories rather than the one special case equivalent to GR), one of which—the aforementioned tetrad-teleparallel gauge current—stands out as most suitable for this and most other applications. (v) Instead one could begin directly with homogeneous cosmology: homogeneous cosmologies are naturally described in terms of a preferred homogeneous tetrad; then an energy-momentum expression based on the Hamiltonian for this preferred tetrad is clearly appropriate.

There have been several studies aimed at finding the total energy of the expanding universe. An early investigation proposed that our universe may have arisen as a quantum fluctuation of the vacuum. That model predicted a universe which is homogeneous, isotropic and closed, and consists equally of matter and anti-matter. Albrow [22] and Tryon [23] proposed that our universe must have a zero net value for all conserved quantities and presented some arguments, using a Newtonian order of magnitude estimate, favoring the proposal that the net energy of our universe may indeed be zero. The general argument for this requirement is that energy can always be represented by an integral over a closed 2-surface bounding the region of interest, so if the universe has an empty boundary then the energy should vanish (for a more detailed discussion of the vanishing of energy for a closed cosmology see [24]). Thus for closed cosmological models the total energy is necessarily zero. Years ago Misner [25] pointed out a technical problem with the attempts at explicit demonstrations of this statement. He noted that the integrands that were being used were reference frame dependent (holonomic) pseudotensors and their associated superpotentials. None of those discus-

sions had specifically established exactly how these particular non-tensorial objects behaved under changes of coordinates—but such changes were actually necessary in the calculations since the whole universe could not be covered with one coordinate patch. According to our understanding it was Wallner [26] who first gave for GR a clear demonstration of this important vanishing energy for closed universe requirement; his integrand (effectively the same tetrad-teleparallel gauge current that we will use) was given in terms of a globally defined frame field.

The subject of the total energy of expanding universe models was re-opened by Cooperstock and Israelit [27], Rosen [28], Garecki [29], Jhori *et al.* [30], Feng and Duan [31], and others using various GR energy-momentum definitions. In one of these investigations the Einstein energy-momentum pseudotensor was used to represent the gravitational energy [28], which led to the result that the total energy of a closed Friedman-Lemaître-Robertson-Walker (FLRW) universe is zero. In another, the symmetric pseudotensor of Landau-Lifshitz was used [30]. Some works calculated the total energy of certain anisotropic Bianchi models using different pseudotensors, leading to similar results [32, 33, 34]. More recently Faraoni and Cooperstock [35], in support of Cooperstock's proposal that gravitational energy vanishes in the vacuum, argued (with the aid of a non-minimally coupled scalar source) that the open, or critically open FLRW universes—as well as Bianchi models evolving into de Sitter spacetimes—also have zero total energy. Finally, calculations for the closed FLRW [36] and some anisotropic Bianchi models [37] using the Einstein, Landau-Lifshitz and other complexes in teleparallel gravity also led to vanishing energy.

In the present work, we examine the energy-momentum (for both local and global regions) for a large class of cosmological models—more specifically we consider all 9 Bianchi types of homogeneous cosmological models using the Hamiltonian boundary term quasi-local approach in the context of not only GR but more generally the tetrad-teleparallel theory of gravity [38]. This theory is a generalization of GR (it has been dubbed NGR, for new general relativity); a certain special case which is equivalent to Einstein's theory [17, 39] was first proposed by Møller to solve the energy localization problem. This special case has been referred to as GR_{||}, the teleparallel equivalent to Einstein's theory (a.k.a.TEGR); it has attracted the attention of several investigators, see e.g., [18, 39, 40, 41, 42, 43]. Our motive for this general approach is not only to improve our understanding of these cosmological models, but also to better understand the considered gravity theories and especially to better understand the meaning and application of these quasi-local energy-momentum ideas.

Most of the previous calculations of the energy of the universe used cartesian or spherical coordinate systems; here in contrast we use only the symmetry of spacetime given by the Bianchi group. In this major extension of previous results [36, 37], by using the Hamiltonian for-

malism with *homogeneous boundary conditions* we find that the energy—for *all regions* (any shape, large or small)—*vanishes* for *all* Bianchi class A models and *does not* vanish for any class B model. This result does not depend on the specific type of matter content of the universe (e.g., baryons, radiation, cold dark matter, dark energy, cosmological constant); it depends only on the symmetry of the spacetime given by the structure constants of the Bianchi group. It is noteworthy that we find a simple physical difference related to the division of the Bianchi types into two classes. Moreover, for both GR and the physically viable teleparallel theories, our energy for Bianchi class B turns out to be, contrary to our expectations and presumptions, *negative*.

The outline of the rest of this work is as follows: in the next section we discuss the Bianchi type universes. In Section 3 we briefly discuss the essential elements concerning the conserved energy-momentum expressions for Einstein's general relativity (GR), the general tetrad-teleparallel theory and the teleparallel equivalent of GR. In Section 4 we give the respective Hamiltonian-boundary-term quasi-local expressions. The energy calculation is presented in Section 5. This is followed by our concluding discussion of the remarkable results.

II. BIANCHI TYPE UNIVERSES

Suppose that the four-dimensional spacetime manifold can be foliated by homogeneous space-like hypersurfaces Σ_t of constant time t . Homogeneity means that each spatial hypersurface has a transitive group of isometries. The study of this 3-parameter isometry group, via the classification of 3-dimensional Lie algebras, led to the Bianchi classification of spatially homogeneous universes [44].

These models are characterized by homogeneous—but generally anisotropic—spatial hypersurfaces parameterized by time. In a synchronous coordinate systems, in which the time axis is always normal to the hypersurfaces of homogeneity Σ_t , we take the spacetime orthonormal (co)frame for these cosmological models to have the form

$$\vartheta^0 = dt, \quad \vartheta^a(t) = h^a_k(t)\sigma^k, \quad a = 1, 2, 3 \quad (1)$$

where the three basis one-forms $\sigma^k = \sigma^k(x)$, $k = 1, 2, 3$, depend on spatial position in such a way that

$$d\sigma^k = \frac{1}{2}C^k_{lm}\sigma^l \wedge \sigma^m, \quad (2)$$

with C^k_{lm} being certain constants. The associated spacetime metric then has the form

$$ds^2 = -dt^2 + g_{lm}(t)\sigma^l(x)\sigma^m(x), \quad (3)$$

where $g_{lm} := \delta_{ab}h^a_l h^b_m$ is a spatial 3-metric which depends only upon time; for our analysis it need not be diagonal.

There are 9 Bianchi types distinguished by the particular form of the structure constants C^k_{lm} [44, 45]. They fall into two special classes: class A (types I, II, VI₀, VII₀, VIII, IX) have $A_k := C^m_{km} \equiv 0$ and class B (types III, IV, V, VI_h, VII_h) are characterized by $A_k \neq 0$. For our purposes here we hardly need any more details regarding these types. We note that the respective scalar curvatures are vanishing for Type I, positive for Type IX, and negative for all the other types. Also, although the general idea of these models is homogeneous but non-isotropic, certain special cases can be isotropic; specifically, isotropic Bianchi I, V, IX are, respectively, isometric to the usual Friedmann-Lemaître-Robertson-Walker (FLRW) models: $k = 0, -1, +1$.

III. GR AND THE TETRAD-TELEPARALLEL THEORY

Here we note some features, associated with conserved expressions for energy-momentum, for Einstein's general relativity (GR) and certain alternative tetrad-teleparallel theories, especially the one which is equivalent to GR.

For all cases the analysis can be most conveniently expressed in terms of differential forms and orthonormal (co)frames:

$$\vartheta^\alpha = e^\alpha_i dx^i, \quad i = 0, 1, 2, 3. \quad (4)$$

In such frames the metric coefficients are constant: $g_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$. We will consider on the same space more than one connection (equivalently, more than one covariant derivative or parallel transport rule). However, in all cases the connection is assumed to be metric compatible:

$$0 = dg_{\mu\nu} - \Gamma^\lambda_{\mu} g_{\lambda\nu} - \Gamma^\lambda_{\nu} g_{\mu\lambda} = -\Gamma_{\mu\nu} - \Gamma_{\nu\mu}. \quad (5)$$

Hence the connection one-form coefficients are always antisymmetric: $\Gamma_{\alpha\beta} = \Gamma_{[\alpha\beta]}$. Any particular connection is characterized by its *curvature* (2-form):

$$R^\alpha_{\beta} = d\Gamma^\alpha_{\beta} + \Gamma^\alpha_{\gamma} \wedge \Gamma^\gamma_{\beta}, \quad (6)$$

and its *torsion* (2-form):

$$T^\alpha = d\vartheta^\alpha + \Gamma^\alpha_{\beta} \wedge \vartheta^\beta. \quad (7)$$

A. GR

Einstein's general relativity is based on a Riemannian geometry, using the Levi-Civita connection—which has vanishing torsion. The Einstein field equations can be expressed in the form

$$R^\alpha_{\beta} \wedge \eta_{\alpha}^{\beta\mu} = -2\kappa \mathcal{T}_\mu, \quad (8)$$

where $\eta^{\alpha\beta\cdots} := *(\vartheta^\alpha \wedge \vartheta^\beta \cdots)$ is the dual form basis. The expression on the left is the Einstein 3-form $-2G^\nu_{\mu}\eta_\nu$,

and the quantity on the right is the source energy-momentum 3-form $T^\nu_{\mu}\eta_\nu$, with $\kappa = 8\pi G/c^4$. Here we take units such that $c = 1$.

Let us first note that the above Einstein equation can naturally be rearranged into a certain special form (for similar arguments see [46, 47]):

$$d(\Gamma^\alpha_{\beta} \wedge \eta_{\alpha}^{\beta\mu}) = -\Gamma^\alpha_{\beta} \wedge d\eta_{\alpha}^{\beta\mu} - \Gamma^\alpha_{\gamma} \wedge \Gamma^\gamma_{\beta} \wedge \eta_{\alpha}^{\beta\mu} - 2\kappa \mathcal{T}_\mu. \quad (9)$$

The 2-form *superpotential*, $\Gamma^\alpha_{\beta} \wedge \eta_{\alpha}^{\beta\mu}$, is the key object. Its differential is exact; consequently the 3-form on the right hand side is closed; hence it is *automatically* a conserved current—which includes the material energy-momentum linearly. The remaining pieces on the right hand side (which depend only on geometric quantities) can thus be interpreted as the energy-momentum density of the gravitational field. These forms are just the quantities we wish to use to describe the energy of GR—and this rearrangement of the field equation is the easiest way to get them that we know of.

Nevertheless, here we want to also present some additional material to (i) reinforce our thesis that these indeed are the most suitable energy-momentum expressions for homogeneous GR cosmologies and also to (ii) include a large class of alternate theories.

Hilbert noted that the above Einstein equation (8) can be obtained from the scalar curvature Lagrangian density by regarding it as a function of the metric. Later certain variations on the theme were considered. For the vacuum—or even non-derivative coupled sources like scalar fields or the Maxwell and Yang-Mills gauge fields—one can simply vary the coframe and the metric compatible connection one-form independently. Then the connection variation yields an equation which requires the connection to have vanishing torsion, i.e., to be the Levi-Civita connection of GR. Let us next note a result [48] we can get most simply by proceeding from that Lagrangian formulation.

The scalar curvature Lagrangian 4-form, $\mathcal{L}_R = R\eta = R^{\alpha\beta} \wedge \eta_{\alpha\beta}$, can be rearranged as indicated:

$$R^\alpha_{\beta} \wedge \eta_{\alpha}^{\beta} \equiv (d\Gamma^\alpha_{\beta} + \Gamma^\alpha_{\gamma} \wedge \Gamma^\gamma_{\beta}) \wedge \eta_{\alpha}^{\beta} \equiv d(\Gamma^\alpha_{\beta} \wedge \eta_{\alpha}^{\beta}) + \Gamma^\alpha_{\beta} \wedge d\eta_{\alpha}^{\beta} + \Gamma^\alpha_{\gamma} \wedge \Gamma^\gamma_{\beta} \wedge \eta_{\alpha}^{\beta}. \quad (10)$$

Now by dropping the total differential we obtain

$$2\kappa \mathcal{L}_{\text{tet}1} = -d\vartheta^\mu \wedge \Gamma^\alpha_{\beta} \wedge \eta_{\alpha}^{\beta\mu} + \Gamma^\alpha_{\gamma} \wedge \Gamma^\gamma_{\beta} \wedge \eta_{\alpha}^{\beta\mu}, \quad (11)$$

a modified Lagrangian density which gives the same field equations. It is noteworthy that the canonical momentum conjugate to the co-frame,

$$\tau_\mu := \frac{\partial \mathcal{L}_{\text{tet}1}}{\partial d\vartheta^\mu} = -\Gamma^{\alpha\beta} \wedge \eta_{\alpha\beta\mu} = -\Gamma^{\alpha\beta}_{\gamma} \delta^{\gamma\rho}_{\alpha\beta\mu} \frac{1}{2} \eta_{\tau\rho}, \quad (12)$$

is just the already encountered superpotential 2-form.

Recall that the Levi-Civita connection is linear in the differential of the frame:

$$\Gamma_{\alpha\beta} = \frac{1}{2} [(d\vartheta_\alpha)_{\beta\gamma} + (d\vartheta_\beta)_{\gamma\alpha} + (d\vartheta_\gamma)_{\beta\alpha}] \vartheta^\gamma. \quad (13)$$

This relation can be used to eliminate Γ^α_β from (11) to obtain a certain specific action: $\mathcal{L}_{\text{GRtet}} = \mathcal{L}_{\text{GRtet}}(\vartheta, d\vartheta)$, which is quadratic in $d\vartheta$. In addition to this tetrad version of Einstein's theory there are other more general tetrad theories of gravity which follow from other Lagrangians quadratic in $d\vartheta$. Such tetrad theories are somewhat interesting in their own right. Let us consider them in the next subsection.

B. tetrad-teleparallel theory

In this work we are concerned especially with the general homogeneous Bianchi cosmologies. Such models have a natural preferred global homogeneous orthonormal frame (a.k.a. tetrad, vierbein). If a geometry has a preferred tetrad one can naturally introduce a new parallel transport rule (i.e., a new connection) such that this frame is, by definition, *parallel*. A geometry with a preferred global parallelism is referred to as *teleparallel* (a.k.a. *absolute parallel*). Hence, especially for the homogeneous cosmologies, an appropriate theoretical geometrical framework is the tetrad/teleparallel formulation.

Conversely, for any teleparallel geometry *by definition* the curvature vanishes and the parallel transport is path independent. It follows that a global preferred frame may be constructed by starting with any orthonormal frame at a single point and parallel transporting it along any path to all the other points. The resultant global tetrad field is unique (up to an overall global constant Lorentz transformation). In this constructed frame the connection coefficients vanish. Hence in a teleparallel geometry there is (up to an overall global constant rotation) a *preferred frame* in which the connection also vanishes. The basic variable can then just be taken to be this preferred tetrad, which is most conveniently represented as the (co)frame $\vartheta^\alpha = e^\alpha_i dx^i$. Since the connection coefficients vanish in this frame the associated teleparallel torsion 2-form is then simply given by the frame differential:

$$T^\alpha = \frac{1}{2} T^\mu_{\alpha\beta} \vartheta^\alpha \wedge \vartheta^\beta = d\vartheta^\alpha. \quad (14)$$

Thus we can forgo further mention of the teleparallel connection here and simply regard a teleparallel theory as a theory for a preferred tetrad.

Tetrad theory field equations can be obtained from a Lagrangian 4-form $\mathcal{L}_{\text{tot}} = \mathcal{L}_{\text{tet}}(\vartheta^\alpha, d\vartheta^\alpha) + \mathcal{L}_{\text{mat}}(\vartheta)$. The variation with respect to the frame gives

$$\delta \mathcal{L}_{\text{tot}} = \delta d\vartheta^\mu \wedge \frac{\partial \mathcal{L}_{\text{tet}}}{\partial d\vartheta^\mu} + \delta \vartheta^\mu \wedge \left(\frac{\partial \mathcal{L}_{\text{tet}}}{\partial \vartheta^\mu} + \frac{\partial \mathcal{L}_{\text{mat}}}{\partial \vartheta^\mu} \right). \quad (15)$$

This expression has the form

$$\delta \mathcal{L}_{\text{tot}} = \delta d\vartheta^\mu \wedge \tau_\mu + \delta \vartheta^\mu \wedge (-t_\mu - \mathcal{T}_\mu) \equiv d(\delta \vartheta^\mu \wedge \tau_\mu) + \delta \vartheta^\mu \wedge (d\tau_\mu - t_\mu - \mathcal{T}_\mu), \quad (16)$$

which identifies τ_μ as the canonical conjugate field momentum and t_μ and \mathcal{T}_μ as, respectively, the 3-forms of

gravitational energy-momentum and the material source energy-momentum density. Using (16) and Hamilton's principle gives the tetrad field equation in a form reminiscent of that used by Einstein in his search for his gravity field equations [49]:

$$d\tau_\mu = t_\mu + \mathcal{T}_\mu. \quad (17)$$

The right hand side is *naturally* a conserved current 3-form:

$$d(t_\mu + \mathcal{T}_\mu) = 0, \quad (18)$$

(i.e., because the lhs is exact the rhs must be closed). The associated conserved total energy-momentum within a volume V is

$$P_\mu(V) = \int_V t_\mu + \mathcal{T}_\mu = \int_V d\tau_\mu = \oint_{\partial V} \tau_\mu. \quad (19)$$

The tetrad-teleparallel formulation is natural from the *gauge theory* point of view. From that perspective the above expressions are those of the *translational gauge current* (the conserved current associated with spacetime translations according to Noether's first theorem, see, e.g., [18, 39, 40, 41, 42, 47]).

For the tetrad-teleparallel theories the canonically conjugate momentum field,

$$\tau_\mu := \frac{\partial \mathcal{L}}{\partial T^\mu} = \frac{1}{2} \tau_\mu^{\alpha\beta} \eta_{\alpha\beta}, \quad (20)$$

is generally taken (in order to have quasi-linear second order field equations) to be a linear combination,

$$\tau = \kappa^{-1} (a_1 T_{\text{ten}} + a_2 T_{\text{vec}} + a_3 T_{\text{axi}}), \quad (21)$$

of the tensor, vector, and axivector irreducible parts of the teleparallel torsion [50]:

$$T_{\text{vec}}^{\alpha}{}_{\mu\nu} := \frac{2}{3} \delta^\alpha_{[\nu} T^\lambda{}_{\mu]\lambda}, \quad T_{\text{axi}}^{\alpha\mu\nu} := \frac{1}{3!} \delta^\alpha_{\lambda\sigma\kappa} T^{\lambda\sigma\kappa\mu}, \quad (22)$$

$$T_{\text{ten}} := T - T_{\text{vec}} - T_{\text{axi}}.$$

There is thus a 3-parameter class of such theories [38]. The generic theory determines a *preferred frame*. It turns out, however, that one special parameter choice ($4a_3 = a_2 = -2a_1$) is distinguished: it actually has *local Lorentz gauge freedom*. This model, with $a_1 = -1$, has been encountered hereinbefore (12); it is known as GRtet or GR_{||}, the teleparallel equivalent of Einstein's GR (a.k.a. TEGR), and was first proposed by Møller [17] to solve the GR energy localization problem.

IV. QUASI-LOCAL BOUNDARY EXPRESSION

As mentioned in the introduction, for Einstein's GR many energy-momentum expressions—both quasi-local and reference frame dependent pseudotensors—have been proposed. It should be emphasized that, despite much effort and many nice results, there is *no consensus* as to which, if any, is best. The Hamiltonian approach certainly helps. From that perspective the energy-momentum is determined by the boundary term in the

Hamiltonian [14, 51]. Although (at least formally) the formalism allows for an infinite number of Hamiltonian boundary expressions (including all the superpotentials that generate the pseudotensors), the ambiguities have been tamed: each expression has a geometrically and physically clear significance associated with the boundary conditions determined from the variation of the Hamiltonian [47]. Nevertheless there are (at least formally) an infinite number of possible boundary conditions.

A. Tetrad-teleparallel energy-momentum

On the other hand, the situation for the tetrad/teleparallel theory is in sharp contrast. Investigators [17, 18, 26, 38, 39, 40, 41, 42, 52, 53] were led to only two [54] closely related quasi-local boundary term expressions for the energy-momentum within a volume V :

$$P_i(V) := \oint_{\partial V} e^\mu_i \tau_\mu, \quad P_\mu(V) := \oint_{\partial V} \tau_\mu, \quad (23)$$

they are, respectively, the Møller 1961 [17] expression and the *translational gauge current* derived above. Møller had pointed out that his superpotential (which appears here as a 2-form integrand) is *tensorial* (i.e., it transforms homogeneously under a change of coordinates); however its differential,

$$d(e^\mu_i \tau_\mu) = de^\mu_i \wedge \tau_\mu + e^\mu_i d\tau_\mu, \quad (24)$$

the Møller tetrad-teleparallel energy-momentum 3-form, *is not a tensor* with respect to coordinate transformations (as Møller himself noted)—because of the factor de^μ_i . In contrast, it should be emphasized that both the translation gauge current superpotential 2-form τ_μ and its differential, the gauge current 3-form (17), *are true tensors* under changes of *coordinates*. Generically, the tetrad-teleparallel theory has a natural preferred frame (no local frame gauge freedom), then the translational gauge current energy-momentum expressions have no ambiguity at all.

However for the one special case of greatest interest, GR_{tet}, the theory does have local Lorentz gauge freedom. In that case the gauge current expressions do depend on the choice of orthonormal frame, and thus still contain some observer dependent information mixed in with the physical information in the energy-momentum expression. Nevertheless we can regard the gauge current expressions as preferable to any of the pseudotensors or Møller's 1961 expression, since dependence on an orthonormal frame is more physical than dependence on an arbitrary choice of coordinates.

Concerning the ambiguity of the choice of frame for GR_{tet}, it is important to note that the *quasi-local values* depend only on the choice of frame on the boundary, and not on the choice within the interior of the region.

Moreover in the case of interest here (homogeneous cosmologies) there *is* a preferred frame—and thus there is *no ambiguity* at all.

In summary, in the tetrad-teleparallel formulation unlike GR there is no big ambiguity in the choice of expression. Thus our consideration of this general tetrad/teleparallel class of theories yields two benefits: (i) it allows us to get a result of great generality, applying to this whole class of theories, (ii) moreover, when specialized to TEGR, it determines a *unique* preferred energy-momentum expression for GR. This expression, the tetrad-teleparallel gauge current has been regarded as one of the best, perhaps the best, description of the gravitational energy-momentum for GR.

B. The covariant Hamiltonian approach

On the other hand, let us also briefly discuss the alternative of taking the usual Riemannian geometry approach to GR. A covariant Hamiltonian formulation has been developed for general geometric gravity theories. The Hamiltonian which generates the evolution of a spatial region V along the vector N is given by an integral of the form

$$H(N, V) = \int_V N^\mu \mathcal{H}_\mu + \oint_{\partial V} \mathcal{B}(N). \quad (25)$$

It turns out that \mathcal{H}_μ is proportional to field equations and thus has vanishing value (e.g., for GR (8) we have $\mathcal{H}_\mu = -(2\kappa)^{-1} R^\alpha_\beta \wedge \eta_\alpha^\beta{}_\mu - \mathcal{T}_\mu$). The Hamiltonian boundary term controls the Hamiltonian value and the boundary conditions. It has considerable freedom. The analysis led to certain specific quasi-local Hamiltonian boundary term 2-form expressions related to various types of boundary condition choices (e.g., Dirichlet, Neumann) [15, 47, 51] for quite general gravity theories. When specialized to Riemannian GR, it was noted that one of these expressions is singled out, in that it corresponds to boundary conditions imposed on a complete 4-covariant object (the tetrad) and gives the desired Bondi energy flux [16]:

$$2\kappa \mathcal{B}(N) = \Delta \Gamma^\alpha_\beta \wedge i_N \eta_\alpha^\beta + \bar{D}_\beta N^\alpha \Delta \eta_\alpha^\beta. \quad (26)$$

Here N describes a spacetime vector field which selects the components of energy and momentum, $\Delta \Gamma := \Gamma - \bar{\Gamma}$ and $\Delta \eta := \eta - \bar{\eta}$, where $\bar{\Gamma}$ and $\bar{\eta}$ are reference (or ground state) values. (We note that the same energy-momentum expression in its holonomic form was found by Katz, Bičák, and Lynden-Bell. They have extolled its virtues in several works, see in particular [55] and the references therein.)

Appropriate choices for the Homogeneous cosmologies are to take the frame to be the preferred Bianchi frame, the reference connection to have vanishing components in this frame, and for the spacetime displacement vector field N to be homogeneous, i.e., to have constant components in this frame. Consequently the second term

vanishes and the boundary expression reduces to

$$2\kappa\mathcal{B}(N) = \Gamma^{\alpha\beta} \wedge i_N \eta_{\alpha\beta} = N^\mu \Gamma^{\alpha\beta}_\gamma \delta^{\gamma\tau\rho}_{\alpha\beta\mu} \frac{1}{2} \eta_{\tau\rho}. \quad (27)$$

This is, as promised, the same superpotential form encountered earlier (9,12), an expression whose utility has been recognized at least since [19]. This succinct argument shows that the preferred Riemannian GR covariant Hamiltonian quasi-local boundary term coincides in this situation with the GR_\parallel gauge current. The same expression also follows naturally from the Hamiltonian formulation of the tetrad-teleparallel theory [18] and, moreover, from the Hamiltonian boundary term associated with the Witten positive energy proof [19, 20, 21] with homogeneous spinor field.

V. THE BIANCHI ENERGY CALCULATION

As we have argued, one can regard GR as a special case of the tetrad-teleparallel theory, so we consider in detail the latter formulation. The energy-momentum integral over the boundary of a region at a fixed time t is

$$P_\mu(V) := \oint_{\partial V} \tau_\mu = \oint_{\partial V} \frac{1}{2} \tau_\mu^{\alpha\beta} \eta_{\alpha\beta}. \quad (28)$$

In this Bianchi cosmology case the components of $\tau_\mu^{\alpha\beta}$ and $T^\mu_{\alpha\beta}$ (in the preferred teleparallel frame) are functions of time alone—dependence on the spatial coordinates shows up only in the teleparallel coframe ϑ^μ via the σ^m . Consequently, in detail

$$\begin{aligned} P_\mu(V) &= \frac{1}{2} \tau_\mu^{\alpha\beta}(t) \oint_{\partial V} \eta_{\alpha\beta} = \frac{1}{2} \tau_\mu^{\alpha\beta}(t) \int_V d\eta_{\alpha\beta} \quad (29) \\ &= \frac{1}{2} \tau_\mu^{\alpha\beta}(t) \int_V d\vartheta^\gamma \wedge \eta_{\alpha\beta\gamma} \\ &= \frac{1}{2} \tau_\mu^{\alpha\beta}(t) \frac{1}{2} T^\gamma_{\lambda\delta}(t) \delta^{\kappa\lambda\delta}_{\alpha\beta\gamma} \int_V \eta_\kappa \\ &= \frac{1}{2} \tau_\mu^{\alpha\beta}(t) [T^0_{\alpha\beta} + T^\gamma_{\gamma\alpha} \delta^0_\beta - T^\gamma_{\gamma\beta} \delta^0_\alpha](t) V. \end{aligned}$$

Note that the energy-momentum is given by an integral over the 2-dimensional boundary of the region, yet it turns out to be simply proportional to the size of the included 3-dimensional volume (not its shape); it is also noteworthy that there is no dependence on the location (this is just as it should be considering homogeneity). Effectively, because of the *homogeneity*, we get a homogeneous energy-momentum density, and a unique *localization* of energy-momentum.

To properly appreciate our results here, one should note that most previous calculations of energy for cosmological models considered the isotropic FLRW models, or one or at most a few Bianchi types (and they then usually confined their results to diagonal metrics). They generally used *holonomic* energy-momentum expressions, which correspond to rather different boundary

conditions. Typically they considered some ball of constant radius around the origin or only the whole space. The expressions were not manifestly homogenous; the results were shape dependent, not simply proportional to volume.

Continuing with our calculation, we need to find the explicit value of the teleparallel torsion (14). From (1) we find

$$T^0 = 0, \quad T^a = d(h^a_k \sigma^k) = \dot{h}^a_k dt \wedge \sigma^k + h^a_k d\sigma^k. \quad (30)$$

using this along with (2) we obtain the torsion tensor components $T^0_{\mu\nu} = 0$ and

$$T^a_{0b} = \dot{h}^a_k h^k_b, \quad T^a_{bc} = h^a_k C^k_{lm} h^l_b h^m_c. \quad (31)$$

From the latter it follows that

$$T^\gamma_{b\gamma} = T^c_{bc} = C^m_{km} h^k_b = A_k h^k_b. \quad (32)$$

Consequently the energy-momentum is

$$\begin{aligned} P_\mu(V) &= \tau_\mu^{\alpha\beta}(t) T^\gamma_{\beta\gamma}(t) \delta^0_\alpha V \\ &= \tau_\mu^{0b}(t) T^c_{bc}(t) V = \tau_\mu^{0b}(t) A_k h^k_b(t) V, \end{aligned} \quad (33)$$

which vanishes for all class A models. (The physical interpretation is that the negative gravitational binding energy density exactly cancels the positive material energy density.)

For class B models we need

$$\begin{aligned} \kappa \tau_\mu^{0b} &= (a_1 T_{\text{ten}} + a_2 T_{\text{vec}} + a_3 T_{\text{axi}})_\mu^{0b} \\ &= [a_1 T + (a_2 - a_1) T_{\text{vec}} + (a_3 - a_1) T_{\text{axi}}]_\mu^{0b}. \end{aligned} \quad (34)$$

For energy we need just the $\mu = 0$ component. As $T^0_{\mu\nu} = 0$ and T_{axi} is totally antisymmetric it reduces to

$$\begin{aligned} \kappa \tau_0^{0b} &= (a_2 - a_1) T_{\text{vec}0}^{0b} = (a_2 - a_1) \frac{1}{3} T^c_c{}^b \\ &= (a_2 - a_1) \frac{1}{3} A_k h^k_a \delta^{ab}. \end{aligned} \quad (35)$$

Hence the energy within any volume V at time t is

$$P^0(V) = g^{00} \tau_0^{0b}(t) A_k h^k_b(t) V = \frac{a_1 - a_2}{3\kappa} A_k A_l g^{kl}(t) V. \quad (36)$$

From various investigations it has been found that certain parameter restrictions should be imposed. In particular for the proper Newtonian limit we must require the so-called *viable* condition [50], $2a_1 + a_2 = 0$. Moreover normalization to the Newtonian limit gives $a_1 = -1$. This leads us to the one parameter teleparallel theory, also known as NGR (New General Relativity [38]). If these parameter conditions are satisfied we have, for all viable cases (including the very special case of GR_\parallel , for which we also have $a_1 + 2a_3 = 0$),

$$E = P^0(V) = -\kappa^{-1} A_k A_l g^{kl}(t) V(t) < 0, \quad (37)$$

i.e., *negative energy* for all regions in class B models. With similar computations these same energy results can be directly verified for GR using the relations (12,13,19,27).

Similarly one could find the explicit value of the “linear momentum” $P_c(V)$ [56]. However, to get this one needs to evaluate

$$\tau_c^{0b} = \kappa^{-1} [a_1 T + (a_2 - a_1) T_{\text{vec}} + (a_3 - a_1) T_{\text{axi}}]_c^{0b}, \quad (38)$$

which is linear (but not so simple) in $T^c_{0b} = \dot{h}^c_j h^j_b$. The calculation is straightforward but not enlightening.

VI. DISCUSSION AND CONCLUSIONS

We have obtained some new insight regarding the energy of homogeneous cosmologies, especially from considering the questions: what energy should be associated with a region of the universe? Does the total energy of a closed universe vanish? Is the energy of an open universe positive? Does the energy of empty flat space vanish?

Specifically, using a natural prescription we found the value of the (quasi-)local energy-momentum for the general tetrad-teleparallel theory (which includes Einstein’s GR as an important special case) for all 9 Bianchi types—with general homogeneous gravitational sources. From the Hamiltonian approach we found that, for comoving observers with *homogeneous* boundary conditions and reference, the energy vanishes for all regions in all Bianchi class A models, and it does not vanish for any class B model. This is the case for the whole 3-parameter class of tetrad-teleparallel theories. According to our measure, the one parameter set of viable teleparallel theories with a good Newtonian limit, which includes the teleparallel equivalent of Einstein’s theory, has *negative* energy for all class B models.

We note that all the cosmologies in the Bianchi class A models can be compactified, so our vanishing energy result is *consistent* with the requirement of vanishing total energy for a closed universe. The class B models cannot in general be compactified, however there are some special exceptions. Nevertheless these exceptions are not counter-examples: while in certain cases the metric geometry can be compactified, this cannot be done in such a way that the *frames* match up to give a globally defined smooth frame (i.e., they are not *globally* homogeneous). Indeed a proof of this [57] used a calculation virtually identical to ours above without noting the energy interpretation.

It is noteworthy that our energy depends only on the symmetry of the spacetime given by the structure constants of the Bianchi group, and that our result holds for *all* types of material sources including *dark matter* and *dark energy* (either as a cosmological constant or as some kind of unusual field like *quintessence* appearing as a part of the energy-momentum tensor).

The results presented here can be specialized to the

few Bianchi models which can be isotropic. For those cases, our *homogenous* results have been compared with those found using a similar approach for the more familiar “isotropic-about-one-point” FLRW formulations [58, 59]. The isotropic Bianchi I is isometric to the flat $k = 0$ model, and both are found to have vanishing energy. The isotropic Bianchi IX is isometric to the $k = +1$ model; the energy of the latter for spherically symmetric regions first increases and then decreases to zero as the radius reaches the antipode so that the volume encompasses the whole close space (thereby satisfying the vanishing energy for any closed space criteria), whereas the Bianchi IX has zero energy for all regions. The open $k = -1$ FLRW model was found to have negative energy, qualitatively but not quantitatively like its isometric Bianchi counterparts V and VII_h. It is remarkable that one special case, with scale factor $a(t) = t$, is actually isometric to empty Minkowski space with the expanding spatial slicing, $t = \sqrt{T^2 - R^2}$. The energy is *negative* for our measure applied to the expanding space but, of course, the energy vanishes for Minkowski with the constant T slicing. Thus the open and closed models offer different “localizations of energy” for the same physical situation, and provide good examples of the effect of different boundary conditions and time evolution vectors.

Although our energy values are obtained as integrals over the boundary of a region, they are not truly *quasi-local*. Properly a quasi-local quantity depends only on the physical data on the boundary of a region. Our values are obtained from integrals of the Hamiltonian boundary term over the boundary, *but* the boundary values, the reference values, and the displacement vector N are selected using the *global* homogeneity. This gives us *homogeneous localizations*. For these models the homogeneity gives a physically meaningful preferred *localization*.

Suppose one were to take a region of a homogeneous cosmology and regard it as an *isolated* gravitating system with the exterior being empty, then far away the isolated system would act like a Newtonian mass point and the asymptotic gravitating field should be nearly static. Then there are strong fundamental arguments (see [60, 61, 62, 63, 64]) that the total effective mass-energy—physically determined by the parameters of a large Kepler orbit but mathematically defined as a certain 2-surface integral at spatial infinity—should be positive (i.e., gravity is attractive), and it should vanish iff the system is empty Minkowski space. In particular an isolated gravitating system would allow energy to be extracted—or if left alone it would spontaneously radiate—until it reached the lowest available energy state. By scaling the lowest energy cannot have a finite negative value. A lower bound of negative infinity allows one to extract an infinite amount of energy, a source of perpetual motion, contrary to a fundamental thermodynamics principle—such a system would never be stable, it would continue to radiate forever. Thus the lowest state should be non-negative. The ground state of geometric gravity is Minkowski space with zero energy.

From this perspective it is reasonable to expect that the *quasi-local energy* determined by a suitable boundary integral at a finite radius, would also be non-negative and would vanish iff the interior were flat Minkowski space; indeed these properties have been regarded as desiderata for any good quasi-local energy (see, e.g., [10, 65]).

The measure of energy used here for homogeneous cosmologies is based on a different concept. Here we regard our cosmological region not as an isolated system but rather as existing in an exterior which is just like the interior. This is the essential significance of our choice of homogeneous boundary conditions, reference and evolution. One consequence is that our cosmological energies do not satisfy the two important aforementioned quasi-local desiderata: indeed for the expressions considered *positivity* need not hold, and *zero energy* iff flat Minkowski space does not hold—in either direction.

Regarding $E = 0$. For any *homogeneous* measure of energy in Bianchi class A models it is certainly quite reasonable to have a vanishing value for all regions, since a homogeneous energy density energy must necessarily vanish at least for all compactifiable regions—and these models can be compactified (this is most obvious for Bianchi I with 3 torus topology identifications): the energy of a closed universe must vanish. This does not conform to the standard *quasi-local* criterion of a unique quasi-local $E = 0$ Minkowski ground state.

The most remarkable feature of our results is that many cosmological models were discovered to have *negative* energy. It had been expected that we would always find *non-negative* energy. One consequence of this strong belief was that a minor sign error was, quite unfortunately, overlooked in some calculations; this led to an *incorrect* claim of *positive energy* for Bianchi B models being reported in some recent conference proceedings [58, 59].

It should be noted that a negative energy value for these open Bianchi cosmologies is not just a peculiar feature due to our choice of expression or model. Indeed the results of calculations (reported in the aforementioned conference proceedings), done using essentially the same principles articulated here, likewise led to *negative energy* for the open FLRW (homogeneous and isotropic) model. Furthermore, according to our simple calculations, most commonly used energy expressions will give the same energy signature for both the open FLRW and Bianchi cosmologies. Yet, remarkably, we have not found any conspicuous reports of this simple fact, although there is some evidence which implicitly supports our energy signature. In particular Bannerjee and Sen [32] evaluated the Einstein pseudotensor for some Bianchi models. Among other results, they reported a non-vanishing energy for finite regions in some class B models (specifically, types III, V, VI), but they did not remark that the sign (easily evaluated) of their calculated energy value for these models is actually negative. The Landau-Lifshitz pseudotensor was used by Jhori *et al.* [30] to calculate the energy for the closed $k = +1$ FLRW model; one can

see from their Eq. (10) that they would have found a negative value for the $k = -1$ model.

The stability argument for non-negative energy is quite compelling—for isolated gravitating systems, which should be settling down to a stable equilibrium state. The cosmological models considered here, however, are quite different in kind. They are inherently *dynamic*, very unlike stable isolated systems. We do not see that there is any fundamental objection to assigning to them a negative energy value.

Moreover, according to the following argument it is quite reasonable that negative spatial curvature models may have negative energy. Consider that a region of positive spatial curvature is like a convex gravitational lens, tending to focus light rays, acting as if it had a positive matter density which attracted the rays. Whereas a negative spatial curvature region acts like a concave lens, causing light rays to be defocused, acting as if it had a negative matter density which repelled light rays. Our definition of energy is actually the value of the Hamiltonian but this can be expected to have some correlation with the effective active gravitational mass of the region.

Identifying a good measure for the energy of a gravitating system has remained an outstanding problem. Here we considered this issue for any region in homogeneous cosmologies. We presented for all models the energy value associated with appropriate boundary conditions as given by our favored covariant Hamiltonian-boundary-term approach. We discovered, in contrast to the expectations, zero energy for some non-flat cases and, most surprisingly, negative energy for many (all Bianchi Class B).

Finally, let us again emphasize that our specific result hinges on the particular chosen measure of energy-momentum. We noted five perspectives which led to the same energy-momentum expression for GR: certainly the chosen expression is a natural one for both homogeneous cosmologies and the teleparallel theories, and it coincides with the $GR_{||}$ tetrad-teleparallel gauge current. Yet our most fundamental argument for distinguishing this energy-momentum expression is via the covariant Hamiltonian approach—which includes *and goes beyond* the Noether analysis [47]. It should be noted that we apply the Hamiltonian analysis to the general theory, and then impose the homogeneous symmetry on the resulting expressions. It has long been known that one cannot in general first impose the symmetry and then do the Hamiltonian analysis. That approach happens to work successfully for Bianchi class A models but does not work for class B models [57].

Our energy-momentum is just the value of the Hamiltonian (which dynamically generates the space-time) with homogeneous boundary conditions, reference and time slicing. We emphasize that our approach is entirely geometric (i.e., coordinate independent). Certainly there are other measures of energy-momentum—and they may have their own particular virtues—but we do not see how there can be any other which is more suitable for all

the homogeneous cosmologies. The results reported here can be considered as a standard measure of the energy-momentum for homogeneous cosmologies.

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